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Finite Difference Scheme for the Zakharov Equation as a Model for Nonlinear Wave-Wave Interaction in Ionic Media

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Abstract: In the paper, the coupled 1D Zakharov Equation (ZE) is considered as the model equation for wave-wave interaction in ionic media. A new six point finite difference scheme, which is equivalent to the multi-symplectic integrator, is derived for the model equations. The numerical simulations are also presented for the model.

Keywords: Zakharov equation, Multi-symplectic scheme; Finite difference scheme, Wave-Wave Interaction,

1. Introduction:

Physically, the wave-wave interaction or the wave collisions are common phenomena in science and engineering for both solitary and non-solitary waves. At the classical level, a set of coupled nonlinear wave equations describing the interaction between high-frequency Langmuir waves and low-frequency ion-acoustic waves were firstly derived by Zakharov [1]. Since then, this system has been the subject of a large number of studies. The system can be derived from a hydrodynamic description of the plasma [2,3]. However, some important effects such as transit-time damping and ion nonlinearities, which are also implied by the fact that the values used for the ion damping have been anomalously large from the point of view of linear ion-acoustic wave dynamics, have been ignored in the Zakharov Equation (ZE). That is to say, the ZE is a simplified model of strong Langmuir turbulence. Thus we have to generalize the ZE by taking more elements into account. Starting from the dynamical plasma equations with the help of relaxed Zakharov simplification assumptions, and through taking use of the time-averaged two-time-scale two-fluid plasma description, the Zakharov Equations are generalized to contain the self-generated magnetic field [4]. The ZE are a set of coupled equations as mentioned in [5]

$$iE_t + E_{xx} - nE = 0,$$
$$n_{tt} - n_{xx} - |E|_{xx}^{2} = 0 \qquad (1)$$

where E is the envelope of the high-frequency electric field, n is the plasma density measured from its equilibrium value. Up to now, there are many methods of constructing exact solutions, for instance, the inverse scattering transform [6], the Hirota method [7], the Backlund method [8], the extended tanh-function method [9], the variable separation approach [10], the Adomian methods decomposition method [11–13] and several other numerical [14-16]. However in their numerical simulations, in order to keep the accuracy, there are many constraints. In the paper, we discretize the system with finite difference schemes to get the numerical simulations of the ZE

2. A difference scheme for ZE system

Considering the ZE system (1) and taking $E = p(x,t) + iq(x,y),\ \eta = \mu(x,t) + i\xi(x,y)$ we get

$$q_t - p_{xx} = (q\xi - p\mu);\quad p_t + q_{xx} = (p\xi + q\mu)$$
$$\mu_{tt} - \mu_{xx} - \left((p)_{xx}^2 + (q)_{xx}^2\right) = 0; \qquad (2)$$
$$\xi_{tt} - \xi_{xx} = 0$$

Introducing the canonical momenta

$$p_x = b, q_x = a, \mu_x = d, \zeta_x = c,$$
$$\mu_t = e,\ \xi_t = f,\ g = \left(p^2\right)_x,\ h = \left(q^2\right)_x \qquad (3)$$

The above system can be written in the following form

$$Kz_t + Lz_x = \nabla_z S(z) \qquad (4)$$

Which is a multi-symplectic in nature with the state variables

$$z = \left(p, q, b, a, \mu, \xi, d, c, e, f, g, h, p^2, q^2\right)^T \in R^{14}$$

The system is multi-symplectic in the sense that K is a skew-symmetric matrix representative of the t direction and L is a skew-symmetric matrix representative of the x direction. S represents a Hamiltonian function, then (2) can be transformed in

$$q_t - p_{xx} = (\xi q - p\mu);\ p_t + q_{xx} = (p\xi + q\mu);$$
$$p_x = b;\ q_x = a, \mu_x = d;\ \xi_x = c;\ \mu_t = e; \qquad (5)$$
$$\xi_t = f, \left(p^2\right)_x = g;\ \left(q^2\right)_x = h;$$
$$\mu_{tt} - \mu_{xx} - \left((p)_{xx}^2 + (q)_{xx}^2\right) = 0;\ \xi_{tt} - \xi_{xx} = 0;$$

and

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$$q_t - b_x = (\xi q - p\mu)\,;\; p_t + a_x = (p\xi + q\mu)\; p_x = b\,;$$

$$q_x = a\,;\; e_t - d_x - (g_x + h_x) = 0\,;\; f_t - c_x = 0 \qquad (6)$$

$$\mu_x = d\,;\; \xi_x = c\,;\; \mu_t = e\,;$$

$$\xi_t = f\,;\; (p^2)_x = g\,;\; (q^2)_x = h$$

So that

$$\nabla_z S(z) = (kp, kq, b, a, sp, sq, d, c, e, f, g, h, p^2, q^2)^T$$

where

$$kp = (\xi q - p\mu)\,;\; kq = (p\xi + q\mu);$$

$$p_x = b\,;\quad q_x = a\,;\quad sp = 0\,;\; sq = 0;$$

$$\mu_x = d\,;\; \xi_x = c\,;\mu_t = e\,;$$

$$\xi_t = f\,;\; (p^2)_x = g\,;(q^2)_x = h$$

and the pair of skew symmetric matrix K and L are

$$K=\begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

$$L=\begin{bmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&-1&-1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

Using midpoint difference scheme to discretize multi-symplectic ZE system, we get

$$\frac{q_{l+1/2}^{n+1} - q_{l+1/2}^{n}}{\Delta t} - \frac{b_{l+1}^{n+1/2} - b_{l}^{n+1/2}}{\Delta x} = \left(\hat{\xi}\hat{q} - \hat{p}\hat{\mu}\right) \qquad (7)$$

$$\frac{p_{l+1/2}^{n+1} - p_{l+1/2}^{n}}{\Delta t} + \frac{a_{l+1}^{n+1/2} - a_{l}^{n+1/2}}{\Delta x} = \left(\hat{p}\hat{\xi} + \hat{q}\hat{\mu}\right) \qquad (8)$$

$$\frac{p_{l+1}^{n+1/2} - p_{l}^{n+1/2}}{\Delta x} = b_{l+1/2}^{n+1/2} \qquad (9)$$

$$\frac{q_{l+1}^{n+1/2} - q_{l}^{n+1/2}}{\Delta x} = a_{l+1/2}^{n+1/2} \qquad (10)$$

$$\frac{e_{l+1/2}^{n+1} - e_{l+1/2}^{n}}{\Delta t} - \frac{d_{l+1}^{n+1/2} - d_{l}^{n+1/2}}{\Delta x} - \left(\frac{g_{l+1}^{n+1/2} - g_{l}^{n+1/2}}{\Delta x} + \frac{h_{l+1}^{n+1/2} - h_{l}^{n+1/2}}{\Delta x}\right) = 0 \qquad (11)$$

$$\frac{f_{l+1/2}^{n+1} - f_{l+1/2}^{n}}{\Delta t} = \frac{c_{l+1}^{n+1/2} - c_{l}^{n+1/2}}{\Delta x} \qquad (12)$$

$$\frac{\mu_{l+1}^{n+1/2} - \mu_{l}^{n+1/2}}{\Delta x} = d_{l+1/2}^{n+1/2} \qquad (13)$$

$$\frac{\xi_{l+1}^{n+1/2} - \xi_{l}^{n+1/2}}{\Delta x} = c_{l+1/2}^{n+1/2} \qquad (14)$$

$$\frac{\mu_{l+1/2}^{n+1} - \mu_{l+1/2}^{n}}{\Delta t} = e_{l+1/2}^{n+1/2} \qquad (15)$$

$$\frac{\xi_{l+1/2}^{n+1} - \xi_{l+1/2}^{n}}{\Delta t} = f_{l+1/2}^{n+1/2} \qquad (16)$$

$$\frac{(p^2)_{l+1/2}^{n+1} - (p^2)_{l+1/2}^{n}}{\Delta x} = g_{l+1/2}^{n+1/2}$$

$$\frac{(q^2)_{l+1/2}^{n+1} - (q^2)_{l+1/2}^{n}}{\Delta x} = h_{l+1/2}^{n+1/2}$$

$$\hat{\mu} = \mu_{l+1/2}^{n+1/2}, \hat{q} = q_{l+1/2}^{n+1/2}, \hat{p} = p_{l+1/2}^{n+1/2}, \hat{\xi} = \xi_{l+1/2}^{n+1/2}$$

From (7) and (9), we eliminate *b*, we get

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$$\frac{q_{l+1/2}^{n+1}-q_{l+1/2}^{n}+q_{l+3/2}^{n+1}-q_{l+3/2}^{n}}{\Delta t}-\frac{2\left(p_{l+2}^{n+1/2}-2p_{l+1}^{n+1/2}+p_{l}^{n+1/2}\right)}{\left(\Delta x\right)^2} \qquad (17)$$

$$=\left(\xi_{l+1/2}^{n+1/2}q_{l+1/2}^{n+1/2}-\mu_{l+1/2}^{n+1/2}p_{l+1/2}^{n+1/2}\right)+\left(\xi_{l+3/2}^{n+1/2}q_{l+3/2}^{n+1/2}-\eta_{l+3/2}^{n+1/2}p_{l+3/2}^{n+1/2}\right)$$

From (8), (10), we eliminate a we get

$$\frac{p_{l+1/2}^{n+1}-p_{l+1/2}^{n}+p_{l+3/2}^{n+1}-p_{l+3/2}^{n}}{\Delta t}+\frac{2\left(q_{l+2}^{n+1/2}-2q_{l+1}^{n+1/2}+q_{l}^{n+1/2}\right)}{\left(\Delta x\right)^2}$$

$$=\left(\xi_{l+1/2}^{n+1/2}p_{l+1/2}^{n+1/2}+\mu_{l+1/2}^{n+1/2}q_{l+1/2}^{n+1/2}\right)+\left(\xi_{l+3/2}^{n+1/2}p_{l+3/2}^{n+1/2}+\mu_{l+3/2}^{n+1/2}q_{l+3/2}^{n+1/2}\right) \qquad (18)$$

Similarly we eliminate e & d, g & h and c & f. So we can get

$$\left(\frac{\mu_{l+1}^{n+2}-2\mu_{l+1}^{n+1}+\mu_{l+1}^{n}}{(\Delta t)^2}\right)-\left(\frac{\mu_{l+2}^{n+1}-2\mu_{l+1}^{n+1}+\mu_{l}^{n+1}}{(\Delta x)^2}\right)-$$

$$\left(\frac{\left(p^2\right)_{l+2}^{n+1}-2\left(p^2\right)_{l+1}^{n+1}+\left(p^2\right)_{l}^{n+1}}{(\Delta x)^2}+\frac{\left(q^2\right)_{l+2}^{n+1}-\left(q^2\right)_{l+1}^{n+1}+\left(q^2\right)_{l}^{n+1}}{(\Delta x)^2}\right)=0 \qquad (19)$$

$$\left(\frac{\xi_{l+1}^{n+2}-2\xi_{l+1}^{n+1}+\xi_{l+1}^{n}}{(\Delta t)^2}\right)-\left(\frac{\xi_{l+2}^{n+1}-2\xi_{l+1}^{n+1}+\xi_{l}^{n+1}}{(\Delta x)^2}\right)=0 \qquad (20)$$

Multiply (18) with i and adding (17) we get

$$-i\frac{\left(E_{l+2}^{*n}+E_{l+2}^{*n+1}\right)-2\left(E_{l+1}^{*n}+E_{l+1}^{*n+1}\right)+\left(E_{l}^{*n}+E_{l}^{*n+1}\right)}{(\Delta x)^2}$$

$$=-\frac{1}{4}\left(\left(E_{l+1/2}^{*\;n}+E_{l+1/2}^{*\;n+1}\right)\left(\eta_{l+1/2}^{*\;n}+\eta_{l+1/2}^{*\;n+1}\right)+\left(E_{l+3/2}^{*\;n}+E_{l+3/2}^{*\;n+1}\right)\left(\eta_{l+3/2}^{*\;n}+\eta_{l+3/2}^{*\;n+1}\right)\right) \qquad (21)$$

Conjugating (21)

$$i\frac{\left(E_{l+2}^{n+1}+2E_{l+1}^{n+1}+E_{l}^{n+1}\right)-\left(E_{l+2}^{n}+2E_{l+1}^{n}+E_{l}^{n}\right)}{2\Delta t}+\frac{\left(E_{l+2}^{n}+E_{l+2}^{n+1}\right)-2\left(E_{l+1}^{n}+E_{l+1}^{n+1}\right)+\left(E_{l}^{n}+E_{l}^{n+1}\right)}{(\Delta x)^2}$$

$$=\frac{1}{4}\left(\left(E_{l+1/2}^{n}+E_{l+1/2}^{n+1}\right)\left(\eta_{l+1/2}^{n}+\eta_{l+1/2}^{n+1}\right)+\left(E_{l+3/2}^{n}+E_{l+3/2}^{n+1}\right)\left(\eta_{l+3/2}^{n}+\eta_{l+31/2}^{n+1}\right)\right) \qquad (22)$$

Multiply (19) with i and adding Eq. (18) then we can get

$$\frac{\left(\eta_{l+1}^{n+2}-2\eta_{l+1}^{n+1}+\eta_{l+1}^{n}\right)}{(\Delta t)^2}-\frac{\left(\eta_{l+2}^{n+1}-2\eta_{l+1}^{n+1}+\eta_{l}^{n+1}\right)}{(\Delta x)^2}-\left(\frac{\left|E_{l+2}^{n+1}\right|^2-2\left|E_{l+1}^{n+1}\right|^2+\left|E_{l}^{n+1}\right|^2}{(\Delta x)^2}\right)=0 \qquad (23)$$

3. Numerical simulation

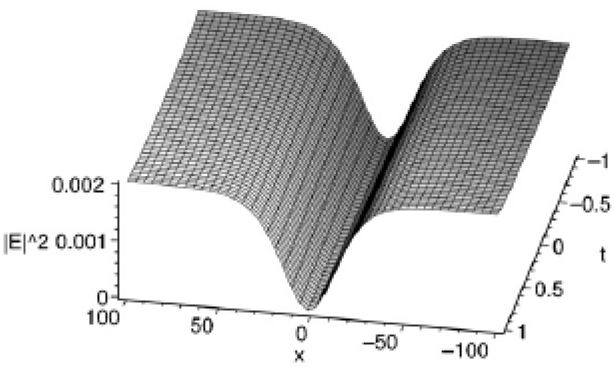
In order to verify numerically whether the proposed methodology leads to higher accuracy, we evaluate the numerical solutions of the ZE (1) with initial conditions

$$E(x,0)=r\tanh(px)\exp[ikx],$$

$$\eta(x,0)=s+r^2\tanh^2(px)/(-4k^2+1)$$

$$\eta_t(x,0)=\frac{-4kpr^2\tanh(px)(1-\tanh^2(px))}{(-4k^2+1)}$$

where $r=\sqrt{\frac{p^2(4k^2-1)}{1+(4k^2-1)\beta}}$, p,k,s,β *are arbitrary constt.*



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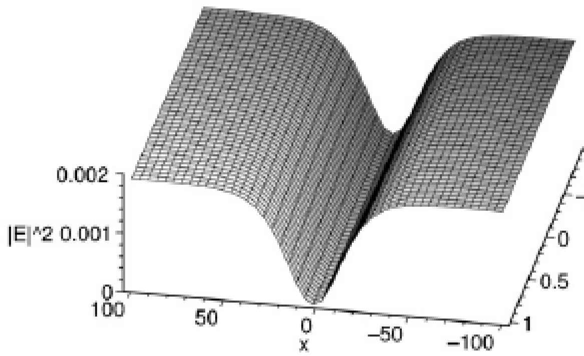

Fig. 1.(a) The numerical result for $|E|^2$ using (22), (b) the exact solution for $|E|^2$ with the initial condition (18) when the parameters *p = 0.05, k = 1, β= 1, s = 0.33.*

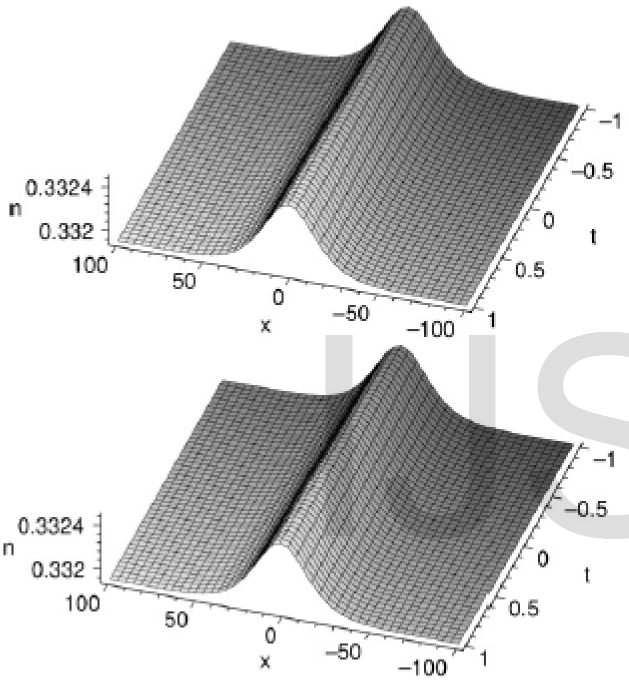

Fig. 1.(a) The numerical result for η using (23), (b) the exact solution for η with the initial condition (18) when the parameters *p = 0.05, k = 1, β = 1, s = 0.33.*

4. Conclusions

A finite Difference Scheme is setup to find the solitary wave solution of the Zakharov Equations. We took some important effects such as transit-time damping and ion nonlinearities into account, there still exist stable solitary wave solutions. The method presented in this paper is only an initial work, more work will be done. It is obvious that the applications of this method to other nonlinear imaginary equations can yield more and more solitary wave solutions.

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